

Generalized ridge estimator and model selection criterion in multivariate linear regression

Yuichi Mori*, Taiji Suzuki*[†]

Abstract

We propose new model selection criteria based on generalized ridge estimators dominating the maximum likelihood estimator under the squared risk and the Kullback-Leibler risk in multivariate linear regression. Our model selection criteria have the following favorite properties: consistency, unbiasedness, uniformly minimum variance. Consistency is proven under an asymptotic structure $\frac{p}{n} \rightarrow c$ where n is the sample size and p is the parameter dimension of the response variables. In particular, our proposed class of estimators dominates the maximum likelihood estimator under the squared risk even when the model does not include the true model. Experimental results show that the risks of our model selection criteria are smaller than the ones based on the maximum likelihood estimator and that our proposed criteria specify the true model under some conditions.

1 Introduction

Model selection criteria such as AIC (Akaike (1971)) and Cp (Mallows (1973)) have been used in various applications and their theoretical properties have been extensively studied. We consider a model selection problem in a multivariate linear regression based on a kind of generalized ridge estimators. The multivariate linear regression considered in this paper has p response variables on a subset of k explanatory variables, and the response is contaminated by a multivariate normal noise. This model, in which the response is multivariate, is thus an extension of multiple linear regression, where the response is univariate. Applications of multivariate linear regression include genetic data analysis, (e.g., Gharagheizi (2008)) and multiple brain scans (e.g., Bassar and Pierpaoli (1998)). Multivariate linear regression is written as

$$Y \sim \mathcal{N}_{n \times p}(AB, \Sigma \otimes I_n),$$

where Y is an $n \times p$ observation matrix of p response variables, A is an $n \times k$ observation matrix of k explanatory variables, B is a $k \times p$ unknown matrix

*Department of Mathematical and Computing Sciences Graduate School of Information Science and Engineering Tokyo Institute of Technology

[†]PRESTO,JST

of regression coefficients, Σ is a $p \times p$ unknown covariance matrix, k is a non-stochastic number, and n is the sample size. We assume that, for all $n \geq k$, $n - p - k - 1 > 0$ and that $\text{rank}(A) = k$.

The purpose of the model selection problem is to select an appropriate subset of regression coefficients. Suppose that J denotes a subset of the index set of coefficients $F = \{1, \dots, k\}$. \mathcal{J} denotes the power set of F , and k_J denotes the number of elements that J contains, that is, $k_J = |J|$. Then, the candidate model corresponding to the subset J can be expressed as

$$Y \sim \mathcal{N}_{n \times p}(A_J B_J, \Sigma \otimes I_n),$$

where A_J is an $n \times k_J$ matrix consisting of the columns of A indexed by the elements of J , and B_J is a $k_J \times p$ unknown matrix of regression coefficients. We assume that the candidate model corresponding to $J_* \in \mathcal{J}$ is the true model.

One way to perform model selection in multivariate linear regression is to apply the well known model selection criteria such as AIC (Akaike (1971)), AICc (Bedrick and Tsai (1994)), Cp (Mallows (1973)), and MCp (Fujikoshi and Satoh (1997)). These criteria are unbiased or asymptotic unbiased estimators of the squared risk and the Kullback Leibler risk that are defined as follows:

$$\begin{aligned} R_S(B, \Sigma, \Phi) &= E \left[\text{tr} \left(\Sigma^{-1} (B - \Phi)^\top A^\top A (B - \Phi) \right) \right], \\ R_{\text{KL}}(B, \Sigma, \hat{f}) &= E_{\tilde{Y}, Y} \left[\log \left(\frac{f(\tilde{Y} | B, \Sigma)}{\hat{f}(\tilde{Y} | Y)} \right) \right], \end{aligned}$$

where Φ is an estimator of B , f is the true probability density of Y , and \hat{f} is a predictive density of \tilde{Y} conditional to Y where \tilde{Y} is the independent copy of Y . Cp and MCp are unbiased estimators of the squared risk of the maximum likelihood estimator, and AIC and AICc are asymptotic unbiased and unbiased estimators, respectively, of the Kullback-Leibler risk of the maximum likelihood estimator. In particular, it is shown that MCp and AICc are uniformly minimum variance unbiased estimators of their corresponding risks (Davies et al. (2006)). One important property of a model selection criterion is consistency, that is, it selects the true model asymptotically in probability. Fujikoshi et al. (2014) showed consistency of AIC, AICc, Cp and MCp under an asymptotic structure $\frac{p}{n} \rightarrow c$ and some conditions in multivariate linear regression, although they are not consistent in usual univariate settings.

Besides model selection criteria corresponding to the maximum likelihood estimator as introduced above, some authors have studied those corresponding to other estimators that might dominate the maximum likelihood estimator. Yanagihara and Satoh (2010) investigated an unbiased estimator of the squared risk of the ridge estimator. They developed a model selection criterion to select the model candidate and the parameter of the ridge estimator simultaneously. Furthermore, although Nagai et al. (2012) proposed the model selection criterion of this type, it is not based on estimators that are rigorously proven to dominate the maximum likelihood estimator.

In this paper, we propose new model selection criteria for multivariate linear regression based on new shrinkage estimators dominating the maximum likelihood estimator under the given risks. In particular, even when the model does not include the true model, our proposed estimator dominates the maximum likelihood estimator under the squared risk. Moreover, our model selection criteria have the following favorite properties: consistency, unbiasedness, and uniformly minimum variance. Furthermore, our model selection criteria have closed forms that are given by modifying AICc and MCp.

We construct a class of Bayes estimators that dominate the maximum likelihood estimator under the risks and have a form of the generalized ridge estimator. The generalized ridge estimator of multivariate linear regression is a class of estimators that can be written as

$$(A_J^\top A_J + P_J K_J P_J^\top)^{-1} A_J^\top Y,$$

where K_J is a $k_J \times k_J$ diagonal matrix and P_J is the $k_J \times k_J$ orthogonal matrix of eigenvectors of $(A_J^\top A_J)^{-1}$. In other words,

$$P_J^\top (A_J^\top A_J)^{-1} P_J = D_J, \quad P_J^\top P_J = I_{k_J},$$

where $D_J = \text{diag}(d_{J,1}, d_{J,2}, \dots, d_{J,k_J})$ and $d_{J,1} \geq d_{J,2} \geq \dots \geq d_{J,k_J}$. Our estimator is related to the generalized Bayes estimator proposed by Maruyama and Strawderman (2005) for linear regression, the Stein type estimator proposed by Konno (1991) for multivariate linear regression, and the generalized Bayes estimator proposed by Tsukuma (2009) for multivariate linear regression. In contrast to these estimators, our estimators enable us to construct closed-form model selection criteria based on them. Moreover, as stated above, it is shown that the criteria have several favorable statistical properties. Since our model selection criteria are based on the generalized ridge estimators dominating the maximum likelihood estimator, it is expected that the risks of our estimators on the models selected by our model selection criteria are smaller than the risks of the maximum likelihood estimator on the models selected by MCp and AICc. We carry out numerical experiments to show the properties of our method.

The contents of this paper are summarized as follows. In Section 2, we list the classes of estimators dominating the maximum likelihood estimator under the squared risk and the Kullback-Leibler risk. Our estimator is given as a Bayes estimator, and eventually, it is shown that it has the same form as a generalized ridge estimator. By setting the hyper parameters of our Bayes estimator appropriately, we derive the class of estimators dominating the maximum likelihood estimator under the squared risk and the Kullback-Leibler risk. In Section 3, we construct model selection criteria based on the estimators in the classes proposed in Section 2. It is also shown that our model selection criteria are uniformly minimum variance unbiased estimators of the two risks respectively and have consistency. In Section 4, we give numerical comparisons between our model selection criteria with AIC, AICc and MCp. In Section 5, we provide the discussion and conclusions.

2 Generalized ridge estimator

In this section, we construct a class of Bayes estimators that have the form of the generalized ridge estimator and dominate the maximum likelihood estimator under the squared risk and the Kullback-Leibler risk. To derive the estimator, we rotate the coordinate and construct the Bayes estimator that can be considered as a generalized ridge estimator on the coordinate. It is shown that the Bayes estimator with tuned hyper parameters dominates the maximum likelihood estimator under the squared risk and the Kullback-Leibler risk. Then, the Bayes estimator is minimax because the maximum likelihood estimator is minimax optimal with constant risk. In particular, in the case of the squared risk, it is not necessary that the candidate model includes the true model. However, in the case of the Kullback-Leibler risk, this property is shown only on models that include the true model.

First, we give a coordinate transformation on Y to derive the estimator. Let Q_J be an $n \times n$ orthogonal matrix such that

$$Q_J A_J = \begin{pmatrix} D_J^{-1/2} P_J^\top \\ 0 \end{pmatrix}.$$

and let D_{*J} be an $n \times n$ diagonal matrix $\text{diag}(d_{J,1}, d_{J,2}, \dots, d_{J,k_J}, 1, \dots, 1)$. We define random matrices $X_J = (x_{J,1}, \dots, x_{J,k_J})^\top \in R^{k_J \times p}$ and $Z_J = (z_{J,1}, \dots, z_{J,n-k_J})^\top \in R^{(n-k_J) \times p}$ such that

$$\begin{pmatrix} X_J \\ Z_J \end{pmatrix} = D_{*J}^{\frac{1}{2}} Q_J Y.$$

Then (X_J, Z_J) has the joint density given by

$$\begin{aligned} & (2\pi)^{-k_J p} |\Sigma|^{-k_J} \prod_i^{k_J} d_{J,i}^{-p} \exp \left\{ -\frac{1}{2} \text{tr}[\Sigma^{-1} (X_J - \Theta_J)^\top D_J^{-1} (X_J - \Theta_J)] \right\} \\ & \times (2\pi)^{(n-k_J)p} |\Sigma|^{-(n-k_J)} \exp \left\{ -\frac{1}{2} \text{tr}[\Sigma^{-1} S_J] \right\}. \end{aligned}$$

where $\Theta_J = P_J^\top B_J = (\theta_{J,1}, \dots, \theta_{J,k_J})^\top$ and $S_J = Z_J^\top Z_J$. With this transformation, the squared risk can be rewritten as

$$E [\text{tr} \Sigma^{-1} (\Phi_J - \Theta_J)^\top D_J (\Phi_J - \Theta_J)],$$

where Φ_J is an estimator of Θ_J . Note that the maximum likelihood estimator of Θ_J is given by X_J , and thus X_J is minimax optimal. We construct the Bayes estimator of Θ_J so that it dominates the maximum likelihood estimator under the squared risk and the Kullback-Leibler risk.

2.1 Derivation of the estimator

In this subsection, we derive a Bayes estimator that is based on the following prior distribution:

$$\Theta_J | \Sigma \sim \mathcal{N}_{k_J \times p}(0, \Sigma \otimes D_J(\Lambda_J^{-1} C_J - \mathbf{I}_{k_J})) = \pi(\Theta_J | \Sigma), \quad (1)$$

$$\Sigma \sim \pi(\Sigma), \quad (2)$$

where Λ_J and C_J are diagonal matrices: $\Lambda_J = \text{diag}(\lambda_{J,1}, \dots, \lambda_{J,k_J})$, $C_J = \text{diag}(c_{J,1}, \dots, c_{J,k_J})$ ($\lambda_{J,i} > 0$, $c_{J,i} > 0$, $i = 1, \dots, k_J$), and $\pi(\Sigma)$ is any distribution on the positive definite matrices such that $E[\Sigma^{-1} | X_J, S_J]$ and $E[\Sigma^{-1} | X_J, S_J]^{-1}$ exist.

Theorem 1 *The Bayes estimator based on the prior (1) and (2) is given by*

$$\hat{B}_{J,B} = P_J \hat{\Theta}_{J,B} = P_J (\mathbf{I}_{k_J} - \Lambda_J C_J^{-1}) D_J P_J^\top A_J^\top Y$$

and is a generalized ridge estimator.

Proof. To derive the Bayes estimator based on this prior, we calculate a part of exponential of the joint density of $(X_J, Z_J, \Theta_J, \Sigma)$. Let $F_J = (\Lambda_J^{-1} C_J - \mathbf{I}_{k_J})^{-1}$. Then,

$$\begin{aligned} & \text{tr}\{\Sigma^{-1}(X_J - \Theta_J)^\top D_J^{-1}(X_J - \Theta_J)\} + \text{tr}\{\Sigma^{-1}\Theta_J^\top (\Lambda_J^{-1} C_J - \mathbf{I})^{-1} D_J^{-1} \Theta_J\} \\ = & \text{tr}\{\Sigma^{-1}[(X_J - \Theta_J)^\top D_J^{-1}(X_J - \Theta_J) + \Theta_J^\top (\Lambda_J^{-1} C_J - \mathbf{I})^{-1} D_J^{-1} \Theta_J]\} \\ = & \text{tr}\{\Sigma^{-1}[X_J^\top D_J^{-1} X_J - X_J^\top D_J^{-1} \Theta_J - \Theta_J^\top D_J^{-1} X_J + \Theta_J^\top (\mathbf{I} + F_J) D_J^{-1} \Theta_J]\} \\ = & \text{tr}\left\{\Sigma^{-1} \left[(\Theta_J - (\mathbf{I} + F_J)^{-1} X_J)^\top (\mathbf{I} + F_J) D_J^{-1} (\Theta_J - (\mathbf{I} + F_J)^{-1} X_J) \right. \right. \\ & \left. \left. + X_J^\top D_J^{-1} (\mathbf{I} - (\mathbf{I} + F_J)^{-1}) X_J \right] \right\}, \end{aligned}$$

and $(\mathbf{I}_{k_J} + F_J)^{-1}$ is given by

$$\begin{aligned} (\mathbf{I}_{k_J} + F_J)^{-1} &= \left(\mathbf{I}_{k_J} + (C_J \Lambda_J^{-1} - \mathbf{I}_{k_J})^{-1} \right)^{-1} \\ &= \left(\mathbf{I}_{k_J} + \Lambda_J (C_J - \Lambda_J)^{-1} \right)^{-1} \\ &= (C_J (C_J - \Lambda_J)^{-1})^{-1} \\ &= \mathbf{I}_{k_J} - \Lambda_J C_J^{-1}. \end{aligned}$$

Therefore, the joint density of $(X_J, Z_J, \Theta_J, \Sigma)$ is proportional to

$$\begin{aligned} & |\Sigma|^{-\frac{n}{2} - \frac{p}{2}} \exp \left\{ -\frac{1}{2} \text{tr} \left\{ \Sigma^{-1} \left[(\Theta_J - (\mathbf{I} - \Lambda_J C_J^{-1}) X_J)^\top (\mathbf{I} - \Lambda_J C_J^{-1})^{-1} D_J^{-1} \right. \right. \right. \\ & \quad \times \left. \left. (\Theta_J - (\mathbf{I} - \Lambda_J C_J^{-1}) X_J) + X_J^\top D_J^{-1} (\mathbf{I} - (\mathbf{I} - \Lambda_J C_J^{-1})) X_J \right] \right\} \right. \\ & \quad \left. - \frac{1}{2} \text{tr}[\Sigma^{-1} S_J] \right\} \pi(\Sigma). \end{aligned}$$

The Bayes estimator of Θ under the squared risk is obtained by

$$\begin{aligned}
& \frac{\partial}{\partial \Phi_J} \mathbb{E} [\text{tr}\{\Sigma^{-1}(\Theta_J - \Phi_J)^\top D_J^{-a}(\Theta_J - \Phi_J)\} | X_J, S_J] = 0 \\
\Rightarrow & \mathbb{E} [2D_J^{-a}(\Theta_J - \Phi_J)\Sigma^{-1} | X_J, S_J] = 0 \\
\Rightarrow & \mathbb{E} [D_J^{-a}\Theta_J\Sigma^{-1} | X_J, S_J] = \mathbb{E} [D_J^{-a}\Phi_J\Sigma^{-1} | X_J, S_J] \\
\Rightarrow & \mathbb{E} [\Theta_J\Sigma^{-1} | X_J, S_J] = \Phi_J \mathbb{E} [\Sigma^{-1} | X_J, S_J] \\
\Rightarrow & \Phi_J = \mathbb{E} [\Theta_J\Sigma^{-1} | X_J, S_J] \{\mathbb{E} [\Sigma^{-1} | X_J, S_J]\}^{-1} \\
\Rightarrow & \Phi_J = \mathbb{E} [\Theta_J | X_J, S_J].
\end{aligned}$$

Therefore, the Bayes estimator $\hat{\Theta}_{J,B}$ based on this prior is $\mathbb{E} [\Theta_J | X_J, S_J]$. From the joint density of $(X_J, Z_J, \Theta_J, \Sigma)$,

$$\hat{\Theta}_{J,B} = (\mathbf{I}_{k_J} - \Lambda_J C_J^{-1}) X_J.$$

Moreover, the Bayes estimator $\hat{B}_{J,B}$ of B_J based on this prior can be written as

$$\begin{aligned}
\hat{B}_{J,B} &= P_J \hat{\Theta}_{J,B} = P_J (\mathbf{I}_{k_J} - \Lambda_J C_J^{-1}) D_J P_J^\top A_J^\top Y \\
&= P_J (D_J^{-1} (\mathbf{I}_{k_J} - \Lambda_J C_J^{-1})^{-1})^{-1} P_J^\top A_J^\top Y \\
&= P_J (D_J^{-1} (\mathbf{I}_{k_J} + \Lambda_J (C_J - \Lambda_J)^{-1})^{-1} P_J^\top A_J^\top Y \\
&= P_J (D_J^{-1} + D_J^{-1} \Lambda_J (C_J - \Lambda_J)^{-1})^{-1} P_J^\top A_J^\top Y.
\end{aligned}$$

Furthermore, let $K_J = D_J^{-1} \Lambda_J (C_J - \Lambda_J)^{-1}$. Then $\hat{B}_{J,B}$ is regarded as the generalized ridge estimator. \square

The Bayes estimator is close to the multivariate form of Maruyama (2005). While he defined a prior of Λ_J as univariate, in this paper, Λ_J is a hyper parameter. Since his estimator does not have a closed form, we change the prior to construct the Bayes estimator expressed in a closed form. We consider a plug-in predictive density by using this Bayes estimator even for the Kullback-Leibler risk because it is easy to handle.

2.2 Generalized ridge estimator under the squared risk

In this subsection, we set the parameters of the generalized ridge estimator and show that it dominates the maximum likelihood estimator under the squared risk for any candidate model. In many studies, although several generalized ridge estimators have been examined, they assume that the candidate model includes the true model. However, this paper does not make this assumption.

In the following theorem, we give sufficient conditions of the parameters so that the Bayes estimator $\hat{B}_{J,B}$ dominates the maximum likelihood estimator \hat{B}_J under the squared risk.

Theorem 2 *Let $C_J = \text{diag}(x_{J,1}^\top S_F^{-1} x_{J,1}, \dots, x_{J,k_J}^\top S_F^{-1} x_{J,k_J})$ and assume that*

$$(i) \quad 0 < \lambda_{J,i} < \frac{2d_{J,i}(p-2)}{n-p-k_F+3} \quad (i = 1, \dots, k_J),$$

(ii) $p \geq 3$ and $n - p - k_F + 3 > 0$,

(iii) $J_* \in \mathcal{J}$.

Then,

$$R_S(B_{J_*}, \Sigma, \hat{B}_{JB}) < R_S(B_{J_*}, \Sigma, \hat{B}_J) \quad (\forall J \in \mathcal{J}).$$

Moreover $R_S(B_{J_*}, \Sigma, \hat{B}_{JB})$ is minimum at $\lambda_{J,i} = \frac{d_{J,i}(p-2)}{n-p-k_F+3}$ ($i = 1, \dots, k_J$).

We employed $c_{J,i} = x_{J,i}^\top S_F^{-1} x_{J,i}$ ($i = 1, 2, \dots, k_J$) instead of $x_{J,i}^\top S_J^{-1} x_{J,i}$ because of the following reason. If we use S_J and not S_F , we need to assume that the candidate model J includes the true model to show that $\hat{B}_{J,B}$ dominates \hat{B}_J . Moreover, in this case, we do not obtain a closed form of the model selection criterion. However, by employing our definition of C_J , we obtain an unbiased estimator of the risk by slightly modifying MCp.

While the assumption (i) of this theorem gives the condition of Λ_J , we may simply set $\lambda_{J,i} = d_{J,i}(p-2)/(n-k_J-p+3)$ ($i = 1, 2, \dots, k_J$). The assumption (ii) of this theorem means a restriction of the dimension. In particular, $p \geq 3$ is the same condition as that under which Stein's estimator dominates the maximum likelihood estimator with respect to the squared risk. The assumption (iii) is to ensure that the candidate models contain the true model.

Each row of $\hat{B}_{J,B}$ is similar to the Stein's estimator. Furthermore, $\hat{B}_{J,B}$ dominates \hat{B}_J under the squared risk when the model J includes the true model. However, this is not obvious when the model J does not include the true model. Furthermore, from the form of lows of $\hat{B}_{J,B}$ and the fact that Stein's estimator is not admissible, $\hat{B}_{J,B}$ is not admissible. However, we employ this form rather than pursuing a statistically optimal one because of its computational efficiency. In fact, the model selection criteria based on the estimator can be analytically computed as shown later.

PROOF. From the assumption (iii), without loss of generality, we may regard J_* as F . Therefore, let B_{J_*} be B_F and A_{J_*} be A_F . For $\forall J \in \mathcal{J}$,

$$\begin{aligned} R_S(B_{J_*}, \Sigma, \hat{B}_{JB}) &= E \left[\text{tr} \left[\Sigma^{-1} (\hat{B}_{JB} - B_{J_*})^\top (A_F^\top A_F) (\hat{B}_{JB} - B_{J_*}) \right] \right] \\ &= E \left[\text{tr} \left[\Sigma^{-1} (A_J \hat{B}_{JB} - A_F B_{J_*})^\top (A_J \hat{B}_{JB} - A_F B_{J_*}) \right] \right] \\ &= R_S(B_{J_*}, \Sigma, \hat{B}_J) \\ &\quad - 2E \left[\text{tr} \left[\Sigma^{-1} (A_J \hat{B}_J - A_F B_{J_*})^\top A_J P_J \Lambda_J C_J^{-1} P_J^\top \hat{B}_J \right] \right] \\ &\quad + E \left[\text{tr} \left[\Sigma^{-1} \hat{B}_J^\top P_J \Lambda_J C_J^{-1} P_J^\top A_J^\top A_J P_J \Lambda_J C_J^{-1} P_J^\top \hat{B}_J \right] \right]. \end{aligned}$$

Therefore it is sufficient to show that the sum of the second and third terms is non-positive. From the definitions of X_J and S_F ,

$$X_J \sim \mathcal{N}_{k_J \times p}(D_J P_J^\top A_J^\top A_F B_{J_*}, \Sigma \otimes D_J), \quad S_F \sim W_p(n - k_F, I_{k_F}).$$

Let $W_J = D_J^{-\frac{1}{2}} X_J = (w_{J,1}, \dots, w_{J,k_J})^\top$. Then

$$\begin{aligned} W_J &= P_J^\top (A_J^\top A_J)^{\frac{1}{2}} \hat{B}_J \sim \mathcal{N}_{k_J \times p}(\Psi_J, \Sigma \otimes I), \\ x_{J,i}^\top S_F^{-1} x_{J,i} &= d_{J,i} w_{J,i}^\top S_F^{-1} w_{J,i}, \end{aligned}$$

where $\Psi_J = P_J^\top (A_J^\top A_J)^{-\frac{1}{2}} A_J A_F B_{J*} = (\psi_{J,1}, \dots, \psi_{J,k_J})^\top$. From the definition of W_J and the joint density of (X_J, S_J) , the joint density of $w_{J,1}, w_{J,2}, \dots, w_{J,k_J}, Z_F$, and $Y^\top (A_F (A_F^\top A_F)^{-1} A_F^\top - A_J (A_J^\top A_J)^{-1} A_J^\top) Y$ is given by

$$\begin{aligned} & (2\pi)^{-k_J p} |\Sigma|^{-k_J} \prod_{i=1}^{k_J} \exp \left\{ -\frac{1}{2} (w_{J,i} - \psi_{J,i})^\top \Sigma^{-1} (w_{J,i} - \psi_{J,i}) \right\} \\ & \times (2\pi)^{(n-k_J)p} |\Sigma|^{-(n-k_J)} \exp \left\{ -\frac{1}{2} \text{tr}[\Sigma^{-1} S_F] \right. \\ & \quad \left. - \frac{1}{2} \text{tr}[\Sigma^{-1} Y^\top (A_F (A_F^\top A_F)^{-1} A_F^\top - A_J (A_J^\top A_J)^{-1} A_J^\top) Y] \right\}. \end{aligned}$$

From the Fisher-Cochran theorem and the definitions of W_J and S_F , $w_{J,i}$ ($i = 1, 2, \dots, k_J$), S_F , and $Y^\top (A_F (A_F^\top A_F)^{-1} A_F^\top - A_J (A_J^\top A_J)^{-1} A_J^\top) Y$ are independent for a fixed model J . Therefore, from Lemma 2.1 of Kubokawa and Srivas-

tava (2001), the second term is given by

$$\begin{aligned}
& -2\mathbb{E} \left[\text{tr} \left[\Sigma^{-1} \left(A_J \hat{B}_J - A_F B_{J*} \right)^\top A_J P_J \Lambda_J C_J^{-1} P_J^\top \hat{B}_J \right] \right] \\
= & -2\mathbb{E} \left[\text{tr} \left[\Sigma^{-1} \left(A_J^\top A_J \hat{B}_J - A_J^\top A_F B_{J*} \right)^\top P_J \Lambda_J C_J^{-1} P_J^\top \hat{B}_J \right] \right] \\
= & -2\mathbb{E} \left[\text{tr} \left[\Sigma^{-1} \left((A_J^\top A_J)^{\frac{1}{2}} \hat{B}_J - (A_J^\top A_J)^{-\frac{1}{2}} A_J A_F B_{J*} \right)^\top (A_J^\top A_J)^{\frac{1}{2}} \right. \right. \\
& \quad \left. \left. \times P_J \Lambda_J C_J^{-1} P_J^\top (A_J^\top A_J)^{-\frac{1}{2}} (A_J^\top A_J)^{\frac{1}{2}} \hat{B}_J \right] \right] \\
= & -2\mathbb{E} \left[\text{tr} \left[\Sigma^{-1} \left(W_J - P_J^\top (A_J^\top A_J)^{-\frac{1}{2}} A_J A_F B_{J*} \right)^\top D_J^{-\frac{1}{2}} \Lambda_J C_J^{-1} D_J^{\frac{1}{2}} W_J \right] \right] \\
= & -2\mathbb{E} \left[\text{tr} \left[\Sigma^{-1} \left(W_J - P_J^\top (A_J^\top A_J)^{-\frac{1}{2}} A_J A_F B_{J*} \right)^\top \Lambda_J C_J^{-1} W_J \right] \right] \\
= & -2 \sum_{i=1}^{k_J} \mathbb{E} \left[\lambda_{J,i} c_{J,i}^{-1} \left(W_J - P_J^\top (A_J^\top A_J)^{-\frac{1}{2}} A_J A_F B_{J*} \right)_{i \cdot}^\top \Sigma^{-1} w_{J,i} \right] \\
= & -2 \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \\
& \quad \times \mathbb{E} \left[(w_{J,i}^\top S_F^{-1} w_{J,i})^{-1} \left(W_J - P_J^\top (A_J^\top A_J)^{-\frac{1}{2}} A_J A_F B_{J*} \right)_{i \cdot}^\top \Sigma^{-1} w_{J,i} \right] \\
= & -2 \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[\text{tr} \left(\nabla_{i \cdot} (w_{J,i}^\top S_F^{-1} w_{J,i})^{-1} w_{J,i} \right) \right] \\
= & -2 \sum_{i=1}^{k_J} \sum_{j=1}^p \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[\frac{\partial}{\partial W_{J,ij}} (w_{J,i}^\top S_F^{-1} w_{J,i})^{-1} W_{J,ij} \right] \\
= & -2p \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[(w_{J,i}^\top S_F^{-1} w_{J,i})^{-1} \right] \\
& \quad + 4 \sum_{i=1}^{k_J} \sum_{j=1}^p \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[(w_{J,i}^\top S_F^{-1} w_{J,i})^{-2} (w_{J,i}^\top S_F^{-1})_j W_{J,ij} \right] \\
= & -2p \sum_{i=1}^{k_J} \mathbb{E} \left[\lambda_{J,i} c_{J,i}^{-1} \right] + 4 \sum_{i=1}^{k_J} \mathbb{E} \left[\lambda_{J,i} c_{J,i}^{-1} \right] \\
= & -2(p-2) \sum_{i=1}^{k_J} \mathbb{E} \left[\lambda_{J,i} c_{J,i}^{-1} \right].
\end{aligned}$$

Similarly, from the proof of Proposition 2.1 of Kubokawa and Srivastava (2001),

the third term is given by

$$\begin{aligned}
& \mathbb{E} \left[\text{tr} \left(\Sigma^{-1} \hat{B}_J^\top P_J \Lambda_J C_J^{-1} P_J^\top A_J^\top A_J P_J \Lambda_J C_J^{-1} P_J^\top \hat{B}_J \right) \right] \\
&= \mathbb{E} \left[\text{tr} \left(\Sigma^{-1} X_J^\top \Lambda_J C_J^{-1} D_J^{-1} \Lambda_J C_J^{-1} X_J \right) \right] \\
&= \mathbb{E} \left[\text{tr} \left(\Sigma^{-1} W_J^\top \Lambda_J^2 C_J^{-2} W_J \right) \right] \\
&= \sum_{i=1}^{k_J} \lambda_{J,i}^2 d_{J,i}^{-2} \mathbb{E} \left[(w_{J,i}^\top S_F^{-1} w_{J,i})^{-2} w_{J,i}^\top \Sigma^{-1} w_{J,i} \right] \\
&= \sum_{i=1}^{k_J} \lambda_{J,i}^2 d_{J,i}^{-2} \mathbb{E} \left[(w_{J,i}^\top S_F^{-1} w_{J,i})^{-1} \right] \\
&= (n - k_F - p + 3) \sum_{i=1}^{k_J} \mathbb{E} \left[d_{J,i}^{-1} \lambda_{J,i}^2 c_{J,i}^{-1} \right].
\end{aligned}$$

Therefore, from the assumptions (i) and (ii), the sum of the second and third terms is given by

$$\begin{aligned}
& -2(T - 2) \sum_{i=1}^{k_J} \mathbb{E} \left[\lambda_{J,i}^2 c_{J,i}^{-1} \right] + (n - k_F - T + 3) \sum_{i=1}^{k_J} \mathbb{E} \left[d_{J,i}^{-1} \lambda_{J,i}^2 c_{J,i}^{-1} \right] \\
&= \sum_{i=1}^{k_J} \lambda_{J,i} \mathbb{E} \left[c_{J,i}^{-1} \right] \left\{ -2(T - 2) + (n - k_F - T + 3) \lambda_{J,i} d_{J,i}^{-1} \right\} \\
&< 0.
\end{aligned}$$

Therefore, $R_S(B_{J_*}, \Sigma, \hat{B}_{J,B})$ is smaller than $R_S(B_{J_*}, \Sigma, \hat{B}_J)$. Moreover, $R_S(B_{J_*}, \Sigma, \hat{B}_{J,B})$ is minimum at $\lambda_{J,i} = \frac{d_{J,i}(p-2)}{(n-k_J-p+3)}$ ($i = 1, 2, \dots, k_J$) because $R_S(B_{J_*}, \Sigma, \hat{B}_J)$ is constant. \square

2.3 Generalized ridge estimator under the Kullback-Leibler risk

In this subsection, we set the parameters of the generalized ridge estimator and show that it dominates the maximum likelihood estimator under the Kullback-Leibler risk when the candidate model includes the true model. We consider a plug-in predictive density that is obtained by plugging-in estimators to B_J and Σ to construct a model selection criterion which is given in a closed form.

Let $\hat{\Sigma}_J$ be the maximum likelihood estimator of Σ on the model J . Then we obtain the condition of the parameters under which the plug-in predictive density with $\hat{B}_{J,B}$ and $\hat{\Sigma}_J$ dominates the plug-in predictive density with \hat{B}_J and $\hat{\Sigma}_J$ under the Kulback-Leibler risk.

Theorem 3 *Let $C_J = \text{diag}(x_{J,1}^\top S_J^{-1} x_{J,1}, \dots, x_{J,k_J}^\top S_J^{-1} x_{J,k_J})$ and assume that*

- (i) $0 < \lambda_{J,i} < \frac{2d_{J,i}(p-2)}{n-k_J-p+1}$,
- (ii) $p \geq 3$ and $n - p - k_J - 1 > 0$,
- (iii) $J_* \subset J$.

Then,

$$R_{\text{KL}}(B_{J_*}, \Sigma, f(\cdot | \hat{B}_{J,B}, \hat{\Sigma}_J)) < R_{\text{KL}}(B_{J_*}, \Sigma, f(\cdot | \hat{B}_J, \hat{\Sigma}_J)).$$

Moreover, $R_{\text{KL}}(B_{J_*}, \Sigma, f(\cdot | \hat{B}_{J,B}, \hat{\Sigma}_J))$ is minimum at $\lambda_{J,i} = \frac{d_{J,i}(p-2)}{n-k_J-T+1}$ ($i = 1, \dots, k_J$).

In the case of the Kullback-Leibler risk, we exclude the case where the candidate model does not include the true model. The reason is as follows. Our estimator is not of the covariance but the mean. However, the plug-in predictive density depends on not only the mean but also the covariance and thus the predictive performance is affected by the estimation performance of covariance. This makes it difficult to analyze whether the plug-in predictive density dominates the maximum likelihood estimator in the case where $J_* \not\subset J$. Even for this situation, it might be possible to construct a plug-in predictive density that dominates the plug-in predictive density with the maximum likelihood estimator. However, our main purpose is to construct a model selection criterion, and thus, we do not pursue this problem in this paper.

PROOF. From the assumption (iii), without loss of generality, we regard J_* as J . The Kullback-Leibler risk of the plug-in predictive density based on $\hat{B}_{J,B}$ and $\hat{\Sigma}_J$ is given by

$$R_{\text{KL}}(B_J, \Sigma, f(\cdot | \hat{B}_{J,B}, \hat{\Sigma}_J)) = E_{\tilde{Y}, Y} \left[\log f(\tilde{Y} | B_J, \Sigma) \right] - E_{\tilde{Y}, Y} \left[\log f(\tilde{Y} | \hat{B}_{J,B}, \hat{\Sigma}_J) \right].$$

Thus, the first terms depends on only the true distribution and not on the plug-in predictive density. The integrand of the second term can be written as

$$-\frac{n}{2} \log |\hat{\Sigma}_J| - \frac{np}{2} \log 2\pi - \frac{1}{2} \text{tr} \left\{ \hat{\Sigma}_J^{-1} (\tilde{Y} - A_J \hat{B}_{J,B})^\top (\tilde{Y} - A_J \hat{B}_{J,B}) \right\}.$$

The third term of this expression can be written as

$$\begin{aligned} & \text{tr} \left\{ \hat{\Sigma}_J^{-1} (\tilde{Y} - A_J \hat{B}_J)^\top (\tilde{Y} - A_J \hat{B}_J) \right\} \\ & + 2 \text{tr} \left\{ \hat{\Sigma}_J^{-1} (\tilde{Y} - A_J \hat{B}_J)^\top (A_J \hat{B}_J - A_J \hat{B}_{J,B}) \right\} \\ & + \text{tr} \left\{ \hat{\Sigma}_J^{-1} (A_J \hat{B}_{J,B} - A_J \hat{B}_J)^\top (A_J \hat{B}_{J,B} - A_J \hat{B}_J) \right\}. \end{aligned}$$

Similarly, $\log f(\tilde{Y} | \hat{B}_J, \hat{\Sigma}_J)$ can be written as

$$-\frac{n}{2} \log |\hat{\Sigma}_J| - \frac{np}{2} \log 2\pi - \frac{1}{2} \text{tr} \left\{ \hat{\Sigma}_J^{-1} (\tilde{Y} - A_J \hat{B}_J)^\top (\tilde{Y} - A_J \hat{B}_J) \right\}.$$

Therefore,

$$\begin{aligned}
& 2R_{\text{KL}}(B_J, \Sigma, f(\cdot|\hat{B}_{J,B}, \hat{\Sigma}_J)) - 2R_{\text{KL}}(B_J, \Sigma, f(\cdot|\hat{B}_J, \hat{\Sigma}_J)) \\
&= 2\mathbb{E}_{\tilde{Y}, Y} \left[\log f(\tilde{Y}|\hat{B}_J, \hat{\Sigma}_J) - \log f(\tilde{Y}|\hat{B}_{J,B}, \hat{\Sigma}_J) \right] \\
&= 2\mathbb{E}_{\tilde{Y}, Y} \left[\text{tr} \left(\hat{\Sigma}_J^{-1} (\tilde{Y} - A_J \hat{B}_J)^\top (A_J \hat{B}_J - A_J \hat{B}_{J,B}) \right) \right] \\
&\quad + \mathbb{E}_{\tilde{Y}, Y} \left[\text{tr} \left(\hat{\Sigma}_J^{-1} (A_J \hat{B}_{J,B} - A_J \hat{B}_J)^\top (A_J \hat{B}_{J,B} - A_J \hat{B}_J) \right) \right].
\end{aligned}$$

The second term of the last expression above is evaluated by

$$\begin{aligned}
& \text{tr} \left\{ \hat{\Sigma}^{-1} (A_J \hat{B}_{J,B} - A_J B_J)^\top (A_J \hat{B}_{J,B} - A_J B_J) \right\} \\
&= n\mathbb{E} \left[\text{tr} (S_J^{-1} X_J^\top \Lambda_J^2 C_J^{-2} D_J^{-1} X_J) \right] \\
&= n \sum_{i=1}^{k_J} \lambda_{J,i}^2 d_{J,i}^{-1} \mathbb{E} \left[(x_{J,i}^\top S_J^{-1} x_{J,i})^{-1} \right].
\end{aligned}$$

Let $W_J = D_J^{-\frac{1}{2}} X_J = (w_{J,1}, \dots, w_{J,k_J})^\top \sim \mathcal{N}_{k_J \times p}(D_J^{-\frac{1}{2}} \Theta_J, \Sigma \otimes \mathbf{I}_{k_J})$. Then from Lemma 2.1 of Kubokawa and Srivastava (2001), the first term is given by

$$\begin{aligned}
& 2\mathbb{E}_{\tilde{Y}, Y} \left[\text{tr} \left(\hat{\Sigma}_J^{-1} (\tilde{Y} - A_J \hat{B}_J)^\top (A_J \hat{B}_J - A_J \hat{B}_{J,B}) \right) \right] \\
&= 2\mathbb{E}_Y \left[\text{tr} \left(\hat{\Sigma}_J^{-1} (A_J B_J - A_J \hat{B}_J)^\top (A_J \hat{B}_J - A_J \hat{B}_{J,B}) \right) \right] \\
&= 2\mathbb{E}_Y \left[n \text{tr} \left(S_J^{-1} (D_J^{-\frac{1}{2}} \Theta_J - W_J)^\top \Lambda_J C_J^{-1} W_J \right) \right] \\
&= -2\mathbb{E}_Y \left[n \sum_{i=1}^{k_J} \lambda_{J,i} c_{J,i}^{-1} (w_{J,i} - d_{J,i}^{-\frac{1}{2}} \Theta_{J,i})^\top S_J^{-1} w_{J,i} \right] \\
&= -2n \sum_{i=1}^{k_J} \lambda_{J,i} \text{tr} \left\{ \mathbb{E}_Y \left[c_{J,i}^{-1} S_J^{-1} w_{J,i} (w_{J,i} - d_{J,i}^{-\frac{1}{2}} \Theta_{J,i})^\top \right] \right\} \\
&= -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E}_Y \left[\text{tr} \left(\nabla_i \left((w_{J,i}^\top S_J^{-1} w_{J,i})^{-1} S_J^{-1} w_{J,i} \right) \Sigma \right) \right] \\
&= -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E}_Y \left[\sum_{j=1}^p \sum_{k=1}^p \frac{\partial}{\partial W_{J,ij}} \left((w_{J,i}^\top S_J^{-1} w_{J,i})^{-1} (S_J^{-1} w_{J,i})_k \right) \Sigma_{jk} \right] \\
&= -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E}_Y \left[\sum_{j=1}^p \sum_{k=1}^p \left(-2(w_{J,i}^\top S_J^{-1} w_{J,i})^{-2} (w_{J,i} S_J^{-1})_j \Sigma_{jk} (S_J^{-1} w_{J,i})_k \right. \right. \\
&\quad \left. \left. + (w_{J,i}^\top S_J^{-1} w_{J,i})^{-1} (S_J^{-1})_{kj} \Sigma_{jk} \right) \right] \\
&= -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E}_Y \left[-2(w_{J,i}^\top S_J^{-1} w_{J,i})^{-2} w_{J,i}^\top S_J^{-1} \Sigma S_J^{-1} w_{J,i} \right. \\
&\quad \left. + (w_{J,i}^\top S_J^{-1} w_{J,i})^{-1} \text{tr}(S_J^{-1} \Sigma) \right].
\end{aligned}$$

Let $w'_{J,i} = \Sigma^{-\frac{1}{2}} w_{J,i}$, $Z'_J = Z_J \Sigma^{-\frac{1}{2}}$ and $S'_J = Z'^{\top}_J Z'_J$. Then $w'_{J,i} \sim \mathcal{N}_T(d_{J,i}^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} \Theta_{J,i}, \mathbf{I})$, $Z'_J \sim \mathcal{N}_{(n-k_J) \times p}$, and $S'_J \sim \mathcal{W}_p(n-k_J, \mathbf{I})$ where $\mathcal{W}_p(k, \Sigma)$ is the Wishart distribution that has degree of freedom k and scale matrix Σ . Let $R_{J,i}^{(1)}$ be a $p \times p$ orthogonal matrix such that $R_{J,i}^{(1)} w'_{J,i} = (\sqrt{w'^{\top}_{J,i} w'_{J,i}}, 0, \dots, 0)^{\top}$, and let $R_{J,i}^{(1)} Z_J^{\top} = (v_{J,i,1}, v_{J,i,2})^{\top}$ where $v_{J,i,1}$ is an $(n-k_J) \times 1$ vector and $v_{J,i,2}$ is an $(n-k_J) \times (p-1)$ matrix. Furthermore, let $R_{J,i}^{(2)}$ be an $(n-k_J) \times (n-k_J)$ orthogonal matrix such that $R_{J,i}^{(2)} v_{J,i,2} = (0, (v_{J,i,2}^{\top} v_{J,i,2})^{\frac{1}{2}})^{\top}$, and let $R_{J,i}^{(2)} v_{J,i,1} = (u_{J,i,1}^{\top}, u_{J,i,2}^{\top})^{\top}$ where $u_{J,i,1}$ is a $(n-p-k_J+1) \times 1$ vector and $u_{J,i,2}$ is a $(p-1) \times 1$ vector. Then

$$\begin{aligned} w'_{J,i} S_J'^{-1} w'_{J,i} &= \frac{w'^{\top}_{J,i} w'_{J,i}}{v_{J,i,1}^{\top} (\mathbf{I}_{n-k_J} - v_{J,i,2} v_{J,i,2}^{\top})^{-1} v_{J,i,2}^{\top}} v_{J,i,1} \\ &= \frac{w'^{\top}_{J,i} w'_{J,i}}{u_{J,i,1}^{\top} u_{J,i,1}} \end{aligned}$$

and let

$$\begin{aligned} V_{J,i} &= R_{J,i}^{(1)} Z_J^{\top} Z_J^{\top} R_{J,i}^{(1)} = R_{J,i}^{(1)} S_J R_{J,i}^{(1)} \\ &= \begin{pmatrix} V_{J,i,11} & V_{J,i,12} \\ V_{J,i,21} & V_{J,i,22} \end{pmatrix} = \begin{pmatrix} v_{J,i,1}^{\top} v_{J,i,1} & v_{J,i,1}^{\top} v_{J,i,2} \\ v_{J,i,2}^{\top} v_{J,i,1} & v_{J,i,2}^{\top} v_{J,i,2} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} &w'_{J,i} S_J'^{-2} w'_{J,i} \\ &= w'^{\top}_{J,i} w'_{J,i} (1, 0, \dots, 0) V_{J,i}^{-2} (1, 0, \dots, 0)^{\top} \\ &= w'^{\top}_{J,i} w'_{J,i} ((V_{J,i}^{-1})_{11}^2 + (V_{J,i}^{-1})_{12} (V_{J,i,21}^{-1})) \\ &= w'^{\top}_{J,i} w'_{J,i} (V_{J,i,11}^{-2} + V_{J,i,11}^{-2} V_{J,i,12} V_{J,i,22}^{-2} V_{J,i,21}) \\ &= \frac{w'^{\top}_{J,i} w'_{J,i}}{(u_{J,i,1}^{\top} u_{J,i,1})^2} (1 + v_{J,i,1}^{\top} v_{J,i,2} (v_{J,i,2}^{\top} v_{J,i,2})^{-2} v_{J,i,2}^{\top} v_{J,i,1}) \\ &= \frac{w'^{\top}_{J,i} w'_{J,i}}{(u_{J,i,1}^{\top} u_{J,i,1})^2} (1 + u_{J,i,2}^{\top} (v_{J,i,2}^{\top} v_{J,i,2})^{-1} u_{J,i,2}), \end{aligned}$$

where $V_{J,i,11 \cdot 2} = V_{J,i,11} - V_{J,i,12} V_{J,i,22}^{-1} V_{J,i,21}$. From the definitions of $w_{J,i}$, $u_{J,i,1}$,

$u_{J,i,2}$ and $v_{J,i,2}$, they are independent. Therefore,

$$\begin{aligned}
& -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[-2(w_{J,i}^\top S_J^{-1} w_{J,i})^{-2} w_{J,i}^\top S_J^{-1} \Sigma S_J^{-1} w_{J,i} \right. \\
& \quad \left. + (w_{J,i}^\top S_J^{-1} w_{J,i})^{-1} \text{tr}(S_J^{-1} \Sigma) \right] \\
& = -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[-2(w'_{J,i} S_J'^{-1} w'_{J,i})^{-2} w_{J,i}'^\top S_J'^{-2} w'_{J,i} + w_{J,i}'^\top S_J'^{-1} w_{J,i}'^{-1} \text{tr} S_J'^{-1} \right] \\
& = -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[-2 \frac{1}{w_{J,i}'^\top w'_{J,i}} (1 + u_{J,i,2}^\top (v_{J,i,2}^\top v_{J,i,2})^{-1} u_{J,i,2}) \right. \\
& \quad \left. + \frac{u_{J,i,1}^\top u_{J,i,1}}{w_{J,i}'^\top w'_{J,i}} ((u_{J,i,1}^\top u_{J,i,1})^{-1} + \text{tr}(V_{J,i}^{-1})_{22}) \right] \\
& = -2n \sum_{i=1}^{k_J} \lambda_{J,i} d_{J,i}^{-1} \mathbb{E} \left[\frac{1}{w_{J,i}'^\top w'_{J,i}} \right] \\
& \quad \times \left\{ -1 + \mathbb{E} [u_{J,i,1}^\top u_{J,i,1}] \mathbb{E} [\text{tr}(V_{J,i}^{-1})_{22}] - 2 \mathbb{E} [u_{J,i,2}^\top (v_{J,i,2}^\top v_{J,i,2})^{-1} u_{J,i,2}] \right\} \\
& = -2n \sum_{i=1}^{k_J} \frac{\lambda_{J,i} d_{J,i}^{-1}}{n - k_J - p + 1} \mathbb{E} \left[\frac{u_{J,i,1}^\top u_{J,i,1}}{w_{J,i}'^\top w'_{J,i}} \right] \\
& \quad \times \left\{ -1 + \frac{(n - k_J - p + 1)(p - 1)}{n - k_J - p - 1} - \frac{2(p - 1)}{n - k_J - p - 1} \right\} \\
& = -2n \sum_{i=1}^{k_J} \frac{\lambda_{J,i} d_{J,i}^{-1} (p - 2)}{n - k_J - p + 1} \mathbb{E} [(w_{J,i}^\top S_J^{-1} w_{J,i})^{-1}] \\
& = -2n \sum_{i=1}^{k_J} \frac{\lambda_{J,i} (p - 2)}{n - k_J - p + 1} \mathbb{E} [c_{J,i}^{-1}]
\end{aligned}$$

Therefore from the assumptions (i) and (ii), this implies that

$$\begin{aligned}
& 2R_{\text{KL}}(B_J, \Sigma, f(\cdot | \hat{B}_{J,B}, \hat{\Sigma}_J)) - 2R_{\text{KL}}(B_J, \Sigma, f(\cdot | \hat{B}_J, \hat{\Sigma}_J)) \\
& = n \sum_{i=1}^{k_J} \lambda_{J,i} \mathbb{E} [c_{J,i}^{-1}] \left(-\frac{2(p - 2)}{n - k_J - p + 1} + d_{J,i}^{-1} \lambda_{J,i} \right) \\
& < 0.
\end{aligned}$$

Furthermore, it is obvious that $R_{\text{KL}}(B_J, \Sigma, f(\cdot | \hat{B}_{J,B}, \hat{\Sigma}_J))$ is minimum at $\lambda_{J,i} = \frac{d_{J,i}(p-2)}{n-k_J-p+1}$ ($i = 1, \dots, k_J$). \square

3 Model selection criterion

In this section, we construct model selection criteria based on the generalized ridge estimators that are proposed in Section 2. We show that the model selection criteria are unbiased estimators of the risks of the generalized ridge estimators. Moreover, we show that they are uniformly minimum variance unbiased estimators and have consistency.

We consider a noncentrality matrix to show consistency. Let $r_J = k_F - k_J$ and

$$\tilde{\Omega}_J = \Sigma^{-\frac{1}{2}} (A_{J_*} B_{J_*})^\top (A_F (A_F^\top A_F)^{-1} A_F^\top - A_J (A_J^\top A_J)^{-1} A_J^\top) A_{J_*} B_{J_*} \Sigma^{-\frac{1}{2}}.$$

Then, we can express $\tilde{\Omega}_J = \Gamma_J^\top \Gamma_J$ where Γ_J is an $r_J \times p$ matrix because the rank of $\tilde{\Omega}_J$ is at most r_J . Moreover, let

$$\begin{aligned} \Omega_J &:= \Gamma_J \Gamma_J^\top, \\ \Xi_J &:= \frac{1}{np} \Omega_J, \end{aligned}$$

where Ω_J and Ξ_J are $r_J \times r_J$ matrices. We call Ω_J the noncentrality matrix of Y on J and let the rank of Ω_J be denoted by γ_J . We assume that γ_J is independent of n and p . Intuitively, Ω_J represents “magnitude” of model misspecification. Indeed, it holds that

$$\Omega_J = 0 \quad (\forall J \supset J_*) \quad (3)$$

because $A_F (A_F^\top A_F)^{-1} A_F^\top$ and $A_J (A_J^\top A_J)^{-1} A_J^\top$ are projection matrices where the range of $A_F (A_F^\top A_F)^{-1} A_F^\top - A_J (A_J^\top A_J)^{-1} A_J^\top$ is perpendicular to the range of A_{J_*} .

3.1 Modified MCp

In this subsection, we propose a model selection criterion that is a modification of MCp under the squared risk. MCp is the estimator of the squared risk with the maximum likelihood estimator and given by

$$\text{MCp}(J) = (n - k_F - p - 1) \text{tr}(S_F^{-1} S_J) + p(2k_J + p + 1 - n),$$

where $\text{MCp}(J)$ is MCp under a model J . We construct a model selection criterion based on the generalized ridge estimator dominating the maximum likelihood estimator. Let $\Lambda_J = \text{diag}\left(\frac{d_{J,1}(p-2)}{n-k_F-p+3}, \dots, \frac{d_{J,k_J}(p-2)}{n-k_F-p+3}\right)$ and $C_J = \text{diag}\left(x_{J,1}^\top S_F^{-1} x_{J,1}, \dots, x_{J,k_J}^\top S_F^{-1} x_{J,k_J}\right)$. The risk is minimum with this setting. Then

$$R_S(B_{J_*}, \Sigma, \hat{B}_{J,B}) = R_S(B_{J_*}, \Sigma, \hat{B}_J) - (p-2) \text{E} [\text{tr}(\Lambda_J C_J^{-1})]$$

by the proof of Theorem 2. Based on this observation, we propose to use ZMCp as an unbiased estimator of $R_S(B_{J_*}, \Sigma, \hat{B}_{J,B})$ under a model J , which is defined as

$$\begin{aligned} & \text{ZMCp}(J) \\ &= \text{MCp}(J) - (p-2) \text{tr}(\Lambda_J C_J^{-1}) \\ &= (n - k_F - p - 1) \text{tr}(S_F^{-1} S_J) + p(2k_J + p + 1 - n) - (p-2) \text{tr}(\Lambda_J C_J^{-1}). \end{aligned}$$

The model selection criterion has the following properties.

Theorem 4 *ZMCp is a uniformly minimum variance unbiased estimator of the squared risk when $J_* \in \mathcal{J}$.*

Theorem 5 *Assume that*

- (i) $J_* \in \mathcal{J}$,
- (ii) $p \rightarrow \infty$, $n \rightarrow \infty$, $\frac{p}{n} \rightarrow c \in (0, 1)$,
- (iii) If $\forall J \in \mathcal{J}_-$ then $\Omega_J = np\Xi_J = O_p(np)$, $\lim_{\frac{p}{n} \rightarrow c} \Xi_J = \Xi_J^*$, and Ξ_J^* is positive definite,
- (iv) $\forall J \in \mathcal{J} C^\perp, j \in J_*$, $\lim_{\frac{p}{n} \rightarrow c} \Theta_{J,i}^\top \Sigma^{-1} \Theta_{J,i} = \infty$,

then

$$\lim_{\frac{p}{n} \rightarrow c} \mathbb{P} \left(\arg \min_{J \in \mathcal{J}} \text{ZMCp}(J) = J_* \right) = 1.$$

The fact of Theorem 4 is obvious by Section 4 of Davies et al. (2006) because ZMCp is described by complete sufficient statistics. The theorem means that ZMCp is the best unbiased estimator of the squared risk of the generalized ridge estimator.

Theorem 5 is seen as an extension of Fujikoshi et al. (2014). He showed that MCp has consistency under similar conditions to Theorem 5.2 of his paper. The difference between the conditions of our result and his is fourth condition. The assumption (iv) is to ensure that the regression coefficients do not have strong correlation, and hence, we can distinguish the candidate models. His conditions do not contain the assumption (iv). The assumption (iv) is necessary for the consistency of ZMCp because ZMCp cannot select the true model when the true model has strong correlation of regression coefficients.

PROOF. From Fujikoshi et al. (2014), we can express the differences between $\text{MCp}(J)$ and $\text{MCp}(J_*)$ as

$$\begin{aligned} & \text{MCp}(J) - \text{MCp}(J_*) \\ &= \left(1 - \frac{p+1}{n-k} \right) ((n-k) (\text{tr}(L_J M_J^{-1}) - \text{tr}(L_{J_*} M_{J_*}^{-1})) + 2p(k_J - k_{J_*})) \\ & \quad + p(p+1) \left(\frac{2(k_J - k_{J_*})}{n-k} \right), \end{aligned}$$

where

$$\begin{aligned} L_J &\sim \mathcal{W}_{r_J}(p, \mathbf{I}_{r_J}; \Omega_J), \\ M_J &\sim \mathcal{W}_{r_J}(n - k_J - p, \mathbf{I}_{r_J}), \end{aligned}$$

$\mathcal{W}_p(k, \Sigma; \Omega)$ is the noncentral Wishart distribution that has degree of freedom k , scale matrix Σ and noncentral matrix Ω . Moreover, L_J and M_J are independently distributed for a fixed model J (but L_J and $L_{J'}$ (or M_J and $M_{J'}$) for different $J, J' \in \mathcal{J}$ could be depended.). Based on a well-known asymptotic method on Wishart distributions, we can see that under assumptions (ii) and (iii)

$$\lim_{\frac{p}{n} \rightarrow c} \frac{1}{np} L_J = \Xi_J^*, \quad \lim_{\frac{p}{n} \rightarrow c} \frac{1}{n} M_J = (1 - c) \mathbf{I}_{r_J}. \quad (4)$$

In the case of $J_* \subset J$, from (3),

$$\lim_{\frac{p}{n} \rightarrow c} \frac{1}{n} L_J = c \mathbf{I}_{r_J}.$$

Therefore, from (4),

$$\begin{aligned} &\lim_{\frac{p}{n} \rightarrow c} \frac{1}{n} \{ \text{MCp}(J) - \text{MCp}(J_*) \} \\ &= (1 - c) \left(\frac{c}{1 - c} + 2c \right) (k_J - k_{J_*}) + 2c^2 (k_J - k_{J_*}) \\ &= c(k_J - k_{J_*}) > 0 \end{aligned}$$

in probability.

Similarly, in the case of $J_* \not\subset J$, from (3),

$$\begin{aligned} &\lim_{\frac{p}{n} \rightarrow c} \frac{1}{np} \{ \text{MCp}(J) - \text{MCp}(J_*) \} \\ &= (1 - c) \frac{1}{1 - c} \text{tr} \Xi_J^* = \text{tr} \Xi_J^* > 0 \end{aligned}$$

in probability.

Therefore, it is sufficient to show that

$$\lim_{\frac{p}{n} \rightarrow c} \frac{p - 2}{n} \text{tr}(\Lambda_J C_J^{-1}) = 0, \quad \forall J \in \mathcal{J}$$

in probability because

$$\begin{aligned} &\text{ZMCp}(J) - \text{ZMCp}(J_*) \\ &= \text{MCp}(J) - \text{MCp}(J_*) + (p - 2) \{ \text{tr}(\Lambda_{J_*} C_{J_*}^{-1}) - \text{tr}(\Lambda_J C_J^{-1}) \}. \end{aligned}$$

From the definitions of $x_{J,i}$ and Z_F , letting $\eta_{J,i} = (D_J P_J^\top A_J^\top A_{J*} P_{J*} \Theta_{J*})_{i,:}$, where $A_{i,:}$ is the i -th row of A , we have

$$d_{J,i}^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} x_{J,i} \sim N_T(d_{J,i}^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} \eta_{J,i}, \mathbf{I}), \quad Z_F \Sigma^{-\frac{1}{2}} \sim N_{(n-k_F) \times T}(0, \mathbf{I} \otimes \mathbf{I}),$$

$$\Sigma^{-\frac{1}{2}} S_F \Sigma^{-\frac{1}{2}} \sim W_T(n - k_F, \mathbf{I}).$$

Therefore we can bound the magnitude of $\lambda_{J,i} c_{J,i}^{-1}$ as

$$\begin{aligned} & \lambda_{J,i} c_{J,i}^{-1} \\ = & \frac{d_{J,i}(p-2)}{n - k_J - p + 3} (x_{J,i}^\top S_F^{-1} x_{J,i})^{-1} \\ = & \frac{p-2}{n - k_J - p + 3} \frac{n - k_F - p - 1}{p} \left(\frac{d_{J,i}^{-1}(n - k_F - p - 1)}{p} x_{J,i}^\top S_F^{-1} x_{J,i} \right)^{-1} \\ = & \frac{p-2}{n - k_J - p + 3} \frac{n - k_F - p - 1}{p} \frac{\chi_{(n-k_F-p-1)}^2}{n - k_F - p - 1} \frac{p}{\chi_p^2(d_{J,i}^{-1} \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i})} \\ = & \frac{p-2}{n - k_J - p + 3} \frac{n - k_F - p - 1}{d_{J,i}^{-1} \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i} + p} \frac{\chi_{(n-k_F-p-1)}^2}{n - k_F - p - 1} \frac{p + d_{J,i}^{-1} \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i}}{\chi_p^2(d_{J,i}^{-1} \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i})} \\ = & O_p \left(\frac{p}{d_{J,i}^{-1} \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i} + p} \right) \\ = & O_p \left(\frac{p}{n \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i} + p} \right) \end{aligned}$$

because

$$\begin{aligned} & \left(\frac{d_{J,i}^{-1}(n - k_F - p - 1)}{T} x_{J,i}^\top S_F^{-1} x_{J,i} \right)^{-1} \\ \sim & F''(n - k_F - T - 1, T, 0, d_{J,i}^{-1} \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i}), \end{aligned}$$

where $\chi_k^2(\delta)$ is the noncentral chi-squared distribution that has degree of freedom k and non-central parameter δ , in particular, we simply write χ_k^2 for $\chi_k^2(0)$ and call χ_k^2 the chi-squared distribution, and $F''(n_1, n_2, \gamma_1, \gamma_2)$ is the doubly noncentral F distribution that has degree of freedom (n_1, n_2) and non-central parameters (γ_1, γ_2) . From the assumption (iv) and $\eta_{J,i}^\top \Sigma^{-1} \eta_{J,i} = O(1) \times \left(\sum_{i=1}^{k_J} \sum_{j=1}^{k_J} \Theta_{J,i}^\top \Sigma^{-1} \Theta_{J,i} \right)$,

$$\lim_{\frac{p}{n} \rightarrow c} \frac{p-2}{n} \text{tr}(\Lambda_J C_J^{-1}) = \lim_{\frac{p}{n} \rightarrow c} \frac{p-2}{n} \sum_{i=1}^{k_J} \lambda_{J,i} c_{J,i}^{-1} = 0$$

in probability. \square

3.2 Modified AICc

In this subsection, we propose a model selection criterion that is a modification of AICc under the Kullback-Leibler risk. AICc is the estimator of the Kullback-Leibler risk with the maximum likelihood estimator and given by

$$\text{AICc}(J) = n \log \left| \frac{1}{n} S_J \right| + np \log 2\pi + \frac{np(n + k_J)}{n - k_J - p - 1},$$

where $\text{AICc}(J)$ is AICc under a model J . As in MCp, we consider the generalized ridge estimator and construct an unbiased estimator of the Kullback-Leibler risk corresponding to that. $R_{\text{KL}}(B_{J_*}, \Sigma, \hat{f}(\cdot | \hat{B}_{J,B}, \Sigma_J))$ is given by

$$\begin{aligned} & 2R_{\text{KL}}(B_{J_*}, \Sigma, \hat{f}(\cdot | \hat{B}_{J,B}, \Sigma_J)) \\ &= 2R_{\text{KL}}(B_{J_*}, \Sigma, \hat{f}(\cdot | \hat{B}_J, \Sigma_J)) - n \frac{p-2}{n - k_J - p + 1} \text{E} [\text{tr}(\Lambda_J C_J^{-1})] \end{aligned}$$

by the proof of Theorem 3. In contrast to AICc which is the unbiased estimator of $R_{\text{KL}}(B_{J_*}, \Sigma, \hat{f}(\cdot | \hat{B}_J, \Sigma_J))$, we denote by ZKLIC an unbiased estimator of $R_{\text{KL}}(B_{J_*}, \Sigma, \hat{f}(\cdot | \hat{B}_{J,B}, \Sigma_J))$ under a model J which is given by

$$\begin{aligned} & \text{ZKLIC}(J) \\ &= \text{AICc}(J) - n \frac{p-2}{n - k_J - p + 1} \text{tr}(\Lambda_J C_J^{-1}) \\ &= n \log \left| \frac{1}{n} S_J \right| + np \log 2\pi + \frac{np(n + k_J)}{n - k_J - p - 1} - n \frac{p-2}{n - k_J - p + 1} \text{tr}(\Lambda_J C_J^{-1}). \end{aligned}$$

The model selection criterion has the following properties.

Theorem 6 *ZKLIC is a uniformly minimum variance unbiased estimator of the Kullback-Leibler risk when $J_* \subset J$.*

Theorem 7 *Assume that*

- (i) $J_* \in \mathcal{J}$,
- (ii) $p \rightarrow \infty$, $n \rightarrow \infty$, $\frac{p}{n} \rightarrow c \in (0, 1)$,
- (iii) If $J_* \not\subset J$ then $\Omega_J = np\Xi_J = O_p(np)$, $\lim_{\frac{p}{n} \rightarrow c} \Xi_J = \Xi_J^*$ and Ξ_J^* is positive definite,
- (iv) $\forall J \in \mathcal{J} \setminus J_*$, $j \in J_*$, $\lim_{\frac{p}{n} \rightarrow c} \Theta_{J,i}^\top \Sigma^{-1} \Theta_{J,i} = \infty$,

then

$$\lim_{\frac{p}{n} \rightarrow c} \text{P} \left(\arg \min_{J \in \mathcal{J}} \text{ZKLIC}(J) = J_* \right) = 1.$$

The assumption (iv) is made for the same reason as discussed in Theorem 5. We again observe that ZKLIC has consistency like ZMCp.

PROOF. From the proof of Theorem 2 of Fujikoshi et al. (2012),

$$\lim_{\frac{p}{n} \rightarrow c} \frac{1}{n \log p} \{\text{AICc}(J) - \text{AICc}(J_*)\} = \gamma_J > 0, \quad J_* \not\subset J,$$

$$\lim_{\frac{p}{n} \rightarrow c} \frac{1}{p} \{\text{AICc}(J) - \text{AICc}(J_*)\} = r_J \left\{ \frac{1}{c} \log(1-c) + 2 \right\} > 0, \quad J_* \subset J, \quad J \neq J_*.$$

Therefore, it is sufficient to show that

$$\lim_{\frac{p}{n} \rightarrow c} \frac{1}{n \log p} \frac{n(p-2)}{n - k_J - T + 1} \text{tr}(\Lambda_J C_J^{-1}) = 0, \quad J_* \not\subset J,$$

$$\lim_{\frac{p}{n} \rightarrow c} \frac{1}{p} \frac{n(p-2)}{n - k_J - T + 1} \text{tr}(\Lambda_J C_J^{-1}) = 0, \quad J_* \subset J, \quad J \neq J_* \quad (5)$$

in probability because

$$\text{ZKLIC} = \text{AICc} - \frac{n(p-2)}{n - k_J - p + 1} \text{tr}(\Lambda_J C_J^{-1}).$$

In the case of $J_* \subset J$, $J \neq J_*$, (5) can be easily shown by the assumption (iv) and the proof of Theorem 5.

Let $J_* \not\subset J$ then

$$\left(\frac{d_{J,i}^{-1}(n - k_J - p - 1)}{p} x_{J,i}^\top S_J^{-1} x_{J,i} \right)^{-1} \sim F''(n - k_J - p - 1, p, 0, d_{J,i}^{-1} \eta_{J,i}^\top \Sigma^{-1} \eta_{J,i})$$

because

$$d_{J,i}^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} x_{J,i} \sim N_p(d_{J,i}^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} \eta_{J,i}, \mathbf{I}), \quad \Sigma^{-\frac{1}{2}} S_J \Sigma^{-\frac{1}{2}} \sim W_p(n - k_J, \mathbf{I}, \Omega_J).$$

Therefore, from the assumption (iv)

$$\lim_{\frac{p}{n} \rightarrow c} \frac{n}{n \log p} \frac{p-2}{n - k_J - p + 1} \text{tr}(\Lambda_J C_J^{-1}) = 0$$

in probability. \square

4 Numerical study

In this section, we numerically examine the validity of our propositions. The risk of a selected model and the probability of selecting the true model by MCp, ZMCp, AIC, AICc, and ZKLIC were evaluated by Monte Carlo simulations with 1,000 iterations. The ten candidate models $J_\alpha = \{1, \dots, \alpha\}$ ($\alpha = 1, \dots, 10$)

were evaluated. In the experiment to evaluate the risks of the selected models, we employed $n = 100, 200$ and $p/n = 0.04, 0.06, \dots, 0.8$. In the experiment to evaluate the probability of selecting the true model, we employed $n = 100, 200, 400, 600$ and $p/n = 0.04, 0.06, \dots, 0.8$. The true model was determined by $B_{J_*} = (1, -2, 3, -4, 5)^\top 1_p^\top$, $J_* = \{1, 2, 3, 4, 5\}$, and the (a, b) -th element of Σ was defined by $(0.8)^{|a-b|}$ ($a = 1, \dots, p; b = 1, \dots, p$). Here, 1_p is the p -dimensional vector of ones. Thus, J_1, J_2, J_3 , and J_4 are underspecified models, and J_6, J_7, J_8, J_9 , and J_{10} are overspecified models. Explanatory variables A is generated in two different ways: in Case 1, $A_{a,b} = u_a^{(b-1)}$ ($a = 1, \dots, n; b = 1, \dots, 10$), where $u_1, \dots, u_n \sim U(-1, 1)$ i.i.d, and in Case 2, $A_{a,b} \sim N(0, 1)$ i.i.d..

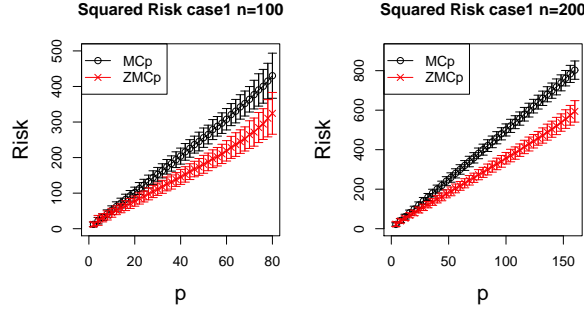


Figure 1: Comparison between MCp and ZMCp under the squared risk in case 1.

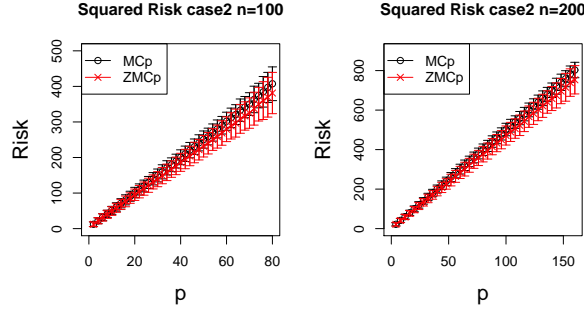


Figure 2: Comparison between MCp and ZMCp under the squared risk in case 2.

Figure 1 and Figure 2 show the squared risks of the selected models by MCp and ZMCp. While Figure 3 and Figure 4 show the Kullback-Leibler risks of

the selected models by AIC, AICc, and ZKLIC, the constant part depending on the true distribution was subtracted. In Case 1, it is seen that ZMCp largely improves MCp. On the other hand, in Case 2, the difference between the squared risks of the selected models is not as much as that in Case 1. The reason is that the explanatory variables have larger correlation in Case 1 than in Case 2, and the generalized ridge estimator is more robust against the correlation.

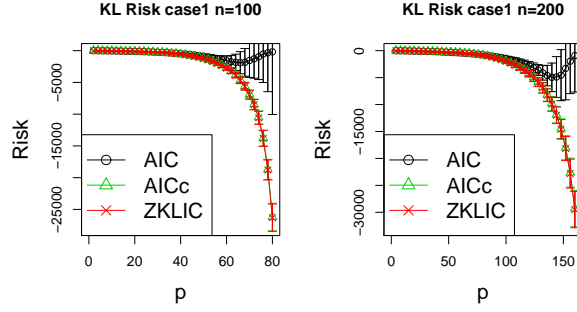


Figure 3: Comparison between AIC, AICc and ZKLIC under the Kullback-Leibler risk in case 1.

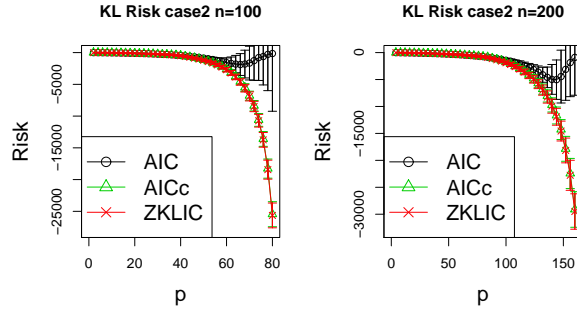


Figure 4: Comparison between AIC, AICc and ZKLIC under the Kullback-Leibler risk in case 2.

Figure 5 and Figure 6 show the probability of selecting the true model by each model selection criterion. In Figure 6, the probability of selecting the true model by our model selection criteria is large when the sample size is large. However, in Figure 5, this probability is small; the matrix of regression coefficient has large correlation. Furthermore, the probabilities of selecting the true model by our model selection criteria are smaller than those of existing

ones in each case. The reason for this is that the variance of our model selection criteria is bigger than those of the existing ones.

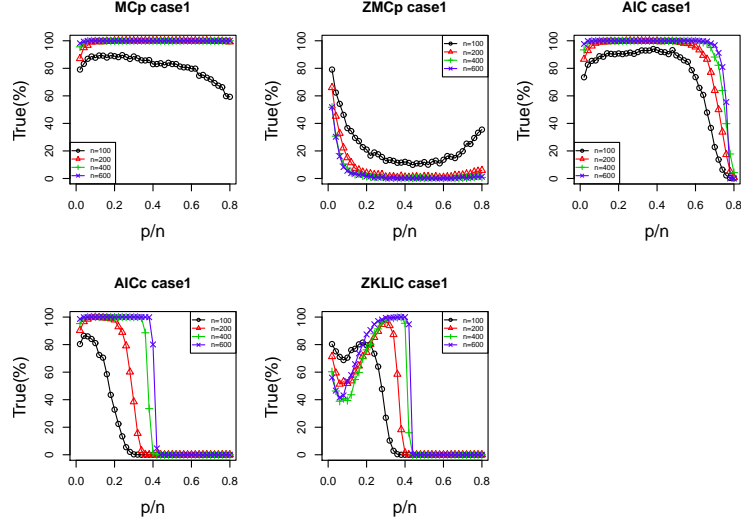


Figure 5: Comparison between MCp, ZMCp, AIC, AICc, and ZKLIC of the probability of selecting the true model in case 1

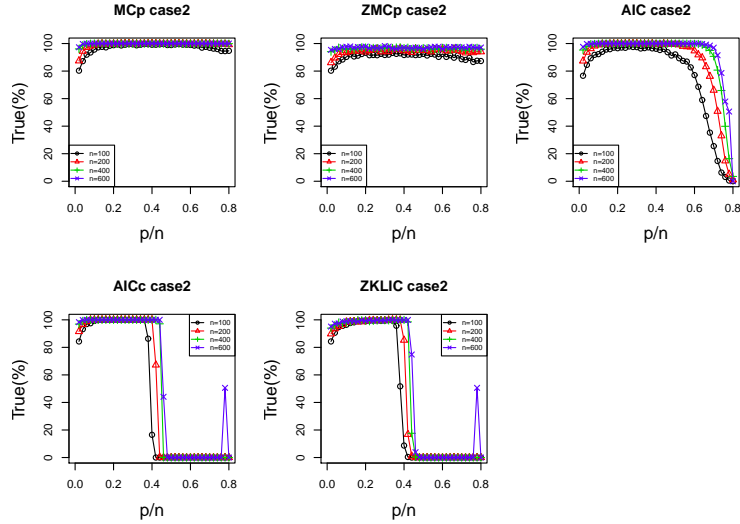


Figure 6: Comparison between MCp, ZMCp, AIC, AICc, and ZKLIC of the probability of selecting the true model in case 2

Although the risks of our model selection criteria are smaller than the ones based on the maximum likelihood estimator, the probability of selecting the true model by our criteria is worse than their probabilities. This is because predictive efficiency and consistency are not compatible (Yang (2005)) and our criteria are specialized in making the risk smaller.

5 Conclusion

In this paper, we proposed model selection criteria based on the generalized ridge estimator, which improves the maximum likelihood estimator under the squared risk and the Kullback-Leibler risk, in multivariate linear regression. Moreover, we showed that our model selection criteria have the same properties as MCp, AIC, and AICc in a high-dimensional asymptotic framework. We demonstrated through the numerical experiments that our model selection criteria have better performances in terms of the risks than the ones based on the maximum likelihood estimators, especially when the matrix of regression coefficients has strong correlation.

Acknowledgement

This work was partially supported by MEXT kakenhi (25730013, 25120012, 26280009, 15H01678 and 15H05707), JST-PRESTO and JST-CREST.

References

- Akaike, H. (1971). Information theory and an extension of the maximum likelihood principle; 1973. *Tsahkadsor, Armenian SSR*, pages 267–281.
- Basser, P. J. and Pierpaoli, C. (1998). A simplified method to measure the diffusion tensor from seven MR images. *Magnetic resonance in medicine*, 39(6):928–934.
- Bedrick, E. J. and Tsai, C.-L. (1994). Model selection for multivariate regression in small samples. *Biometrics*, pages 226–231.
- Davies, S. L., Neath, A. A., and Cavanaugh, J. E. (2006). Estimation optimality of corrected AIC and modified Cp in linear regression. *International statistical review*, 74(2):161–168.
- Fujikoshi, Y., Sakurai, T., and Yanagihara, H. (2012). High-dimensional AIC and consistency properties of several criteria in multivariate linear regression. Technical report, TR.
- Fujikoshi, Y., Sakurai, T., and Yanagihara, H. (2014). Consistency of high-dimensional AIC-type and Cp-type criteria in multivariate linear regression. *Journal of Multivariate Analysis*, 123:184–200.
- Fujikoshi, Y. and Satoh, K. (1997). Modified AIC and Cp in multivariate linear regression. *Biometrika*, 84(3):707–716.
- Gharagheizi, F. (2008). Qspr studies for solubility parameter by means of genetic algorithm-based multivariate linear regression and generalized regression neural network. *QSAR & Combinatorial Science*, 27(2):165–170.
- Konno, Y. (1991). On estimation of a matrix of normal means with unknown covariance matrix. *Journal of Multivariate Analysis*, 36(1):44–55.
- Kubokawa, T. and Srivastava, M. S. (2001). Robust improvement in estimation of a mean matrix in an elliptically contoured distribution. *Journal of multivariate analysis*, 76(1):138–152.
- Mallows, C. L. (1973). Some comments on Cp. *Technometrics*, 15(4):661–675.
- Maruyama, Y. and Strawderman, W. E. (2005). A new class of generalized Bayes minimax ridge regression estimators. *The Annals of Statistics*, 33(4):1753–1770.
- Nagai, I., Yanagihara, H., Satoh, K., et al. (2012). Optimization of ridge parameters in multivariate generalized ridge regression by plug-in methods. *Hiroshima Mathematical Journal*, 42(3):301–324.
- Tsukuma, H. (2009). Generalized Bayes minimax estimation of the normal mean matrix with unknown covariance matrix. *Journal of Multivariate Analysis*, 100(10):2296–2304.

- Yanagihara, H. and Satoh, K. (2010). An unbiased Cp criterion for multivariate ridge regression. *Journal of Multivariate Analysis*, 101(5):1226–1238.
- Yang, Y. (2005). Can the strengths of aic and bic be shared? a conflict between model identification and regression estimation. *Biometrika*, 92(4):937–950.